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tion in the calculus for an engineer. This position is not without a certain element of truth. It is possible to state the integrals necessary to follow an elementary treatment of physics and mechanics in brief space, but a well-grounded knowledge of the subject that will abide with the student is not to be obtained by confining the algebraic work within very narrow limits. It seems to me that with respect to technique neither the extreme view of algebraic formalism that was in vogue twenty-five years ago in the first course in calculus based on certain English textbooks very generally in use in this country nor the practical lack of formalism advocated by those who would popularize the calculus, makes a satisfactory contribution to our picture of the first course. Surely without approximately the amount of technique given in our recent textbooks we should give totally inadequate preparation for the serious study of mechanics, electricity, thermodynamics, rigid dynamics, hydrodynamics, the theory of elasticity, or for the reading of much of the recent literature of physics and chemistry. Indeed, a second course in calculus has been so strongly urged by certain prominent electrical and mechanical engineers that a course has been introduced into the junior year in certain schools to meet the demand. It goes without saying that technique is indispensable to the student who is to become a mathematician and it is clear that we should take due account of this class of men in looking to the future of education along mathematical lines. It seems to me that some of our recent textbooks in the elements of calculus exhibit rare judgment on the part of the authors in their position with respect to the amount of formal work to be done in the first course in calculus.

In conclusion, let me submit that we can make the first course in calculus a more important aid in the appreciation of physical truth by aiming more directly at establishing in the minds of our students what I have tried to indicate as the calculus habit of thinking about phenomena and that this result is most likely to be obtained by creating for the student a mental picture with only a few main lines.

II. The Sign of the Distance in Analytical Geometry.

By Albert A. Bennett, University of Texas.

The pupil in a course in analytical geometry comes very early across two formulas concerning distances. These yield the following schematic equations:

$$[\text{distance from } (x', y') \text{ to } (x, y)]^2 = (x' - x)^2 + (y' - y)^2,$$

$$[\text{distance from } (x', y') \text{ to } ax + by + c = 0]^2 = \frac{(ax' + by' + c)^2}{a^2 + b^2}, \qquad (a^2 + b^2 \neq 0).$$

No discussion of the derivation of these formulas need be made here. The question of present interest is what is to be done in determining the algebraic sign of the distance itself, that is the square root of each of the above expressions. Now the pupil has become accustomed in dealing with square roots to employ conventions of three types with respect to the determination of signs:

1. Numerical determination, where the positive sign only is used.
2. Transcendental determination, where both positive and negative signs occur, but the choice in any given instance is determinate.
3. Algebraic determination, where the sign is indeterminate, and both possibilities as to sign are retained throughout.

The numerical determination is that regularly occurring in elementary arithmetical and geometrical problems, and in many algebraic examples. The transcendental determination is used in the basic conventions of analytical geometry and trigonometry, and the algebraic convention is the rule in algebra except in certain numerical problems. The existence of these distinct conventions is only vaguely present to the pupil's mind but is a frequent source of misconception and error, and merely adds to the student's conviction of the arbitrary and blind character of mathematical notations. But the different points of view are not simultaneously consistent and each has its place, a fact which should at least be appreciated by the teacher.

Each of these three possibilities as applying to the sign of the distance will be treated in turn.

1. **Numerical Determination.** In elementary arithmetic, subtraction is taught, but negative numbers are not usually admitted. A quadratic equation as such does not arise, but the notion of square root is frequently in evidence. One thinks in every case of but a single square root, which is positive. In geometry a construction is given for the mean proportional, and usually mention is not made of two mean proportionals of opposite sign but of one only. Problems in mensuration such as: define the altitude of an equilateral triangle with an integral side, the radius of a circle of given area, etc., lead to square roots, but always to positive quantities. And so the fact that lengths and distances in everyday life are never negative should not surprise the mathematician. The distance from New York to Chicago is equal to the distance from Chicago to New York, the shortest distance from Denver to the Canadian border is equal to the shortest distance from the Canadian border to Denver, etc.

Thus when the student is given an elementary locus problem it might appear natural to regard distances as inherently positive. If a straight line be drawn in a plane, where no reference is made to coördinates, and it be desired to find the locus of points one foot from the line, it seems obvious that the locus will be two lines, not one only. If two points be given, the length of the segment joining them should be the distance between the two points, and the length is inherently positive as usually treated. The locus of points in a plane equally distant from two given intersecting straight lines, each line being viewed as extending indefinitely in both directions, is of course the pair of bisectors of the angles, a pair of perpendicular straight lines. It seems hardly necessary to reinvestigate the question as to whether the center of the inscribed circle of the triangle is equally distant from the sides, and the centers of the escribed circles equidistant from one side and the extensions of the other two sides, or whether the center of the circumscribed circle is equidistant from the vertices. To check up such facts may serve as good pedagogical material but that a convention should retain these relations as still true can occasion no comment.

The numerical determination is, however, subject to the following serious drawback in the elementary treatment of analytical geometry: It requires every algebraic solution to be supplemented by a more or less tedious examination of extraneous solutions for which no adequate preparation is usually made. A few examples will be cited which by their simplicity serve to illustrate sharply the awkwardness inherent in this convention.

1. Find the locus of a point such that the sum of its distances from the rectangular axes is equal to unity.

The locus is a square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$. Here the algebraic work leads to four lines of which only the segments forming the sides of the square satisfy the conditions. The extensions of the sides are extraneous.

2. Find the locus of a point such that the sum of its distances to the two rectangular axes is equal to the distance to the line whose equation is $4x + 3y - 5 = 0$.

The locus is an irregular quadrilateral with vertices on the axes. Only the points of the segments between adjacent vertices satisfy the problem.

3. Find the locus of a point such that the sum of its distances to the two axes and its distance to the line $4x + 3y - 5 = 0$, is unity.

The locus consists of a single point, the origin; all other parts of eight lines are extraneous.

4. Find the locus of a point such that the sum of its distances to the two axes and its distance to the line $4x + 3y - 5 = 0$, is two.

The locus consists of an irregular convex hexagon, whose opposite sides are not equal in length but are parallel.

5. Find the locus of a point such that its distance from $(0, 1)$ exceeds by unity its distance from the X-axis.

The locus consists of one entire parabola tangent to the X-axis at the origin together with the negative half of the Y-axis.

6. Find the locus of a point such that its distance from $(0, 1)$ exceeds by two its distance from the X-axis.

The locus consists of parts only of two parabolas, both having $(0, 1)$ as focus, but with different latera recta, the curves having a common chord on the X-axis and being joined there, but not crossing each other, the portions nearer the focus being extraneous.

7. Find the locus of a point such that the sum of its distances from $(1, 0)$ and $(-1, 0)$ is two.

The locus consists of the segment of the X-axis joining these two points. For the remainder of the X-axis the condition is that the difference of the distances be two, and if the subtrahend and minuend be specified, that one of the portions which is intended is given.

8. Find the locus of a point such that the sum of its distances from the three sides of an equilateral triangle is a given constant.

When the given constant is less than the altitude of the triangle, the locus contains no points.

When the constant is equal to the altitude of the triangle, the locus contains every point within and on, but no point exterior to, the triangle. The locus is in this case not a curve but a portion of the plane.

When the given constant is greater than the altitude of the triangle, the locus consists of a hexagon surrounding the triangle.

In conclusion, the numerical determination while entirely natural and self-consistent, appears to introduce unnecessary complications and non-algebraic investigations not particularly adapted to an elementary course in analytical geometry.

2. **Transcendental Determination.** Analytical geometry became possible by a conscious selection of special conventions facilitating the treatment of negative quantities. Having selected an axis for X's, no algebraic study could determine which direction would be the positive and which the negative. By a non-algebraic and essentially transcendental step one side is adopted as the positive half and the other as the negative. Even when the X-axis is agreed upon, the convention for Y's remains arbitrary. The question of sense is not settled by symmetry. The choice adopted makes the equations for rotation of axes appear in the unsymmetrical form

$$\begin{aligned} x' &= \quad x\cos\theta + y\sin\theta, \\ y' &= -x\sin\theta + y\cos\theta, \end{aligned}$$

rather than in the other possible but equally unsymmetrical form. Seeing what advantages are secured by this arbitrary choice, it is but natural that similar advantages be expected from an arbitrary convention with respect to distances throughout. The entire theory of a coördinate system is to regard given distances as associated with opposite signs when measured in opposite directions, and more than this, to select one direction in each case as positive. Now a line separates the rest of the plane into two parts. If a point be on one side, it cannot move continuously to the other side without passing across the line. What is more obvious than to assign to the distance an algebraic sign, such that one side is reserved for positive distances and the other for negative, these being separated by the line itself whose points are at zero distance? This convention is made in nearly all, if not all, standard American textbooks in analytical geometry. The choice used in the various particular cases is easily suggested by the fact that the origin is always present as a datum point for any line not containing it, and for lines through the origin, a convention consistent with that for the axes is desirable. The convention as to distance between points is usually the numerical determination already mentioned, since otherwise the points of a circle might not all be equally distant from the center.

The transcendental conventions are open to very serious objections, however. For example, the distance of a point from a straight line even if it be selected for convenience as an axis of reference is not independent of the rotation of axes, despite the explicit statements in most of the books concerning the invariance of distance, since the sign changes in certain cases. But consider some selected examples:

1. Find the locus of a point whose distance from the origin is equal to its distance from the Y-axis.

The locus consists of the origin and the positive half only of the axis of X's. Points on the negative half of the X-axis have the opposite sign.

2. Find the locus of a point the sum of whose distances from the two axes is equal to its distance from the origin.

The locus consists of the two positive half axes with the origin.

3. Find the locus of a point at a given constant distance from a straight line.

The result of this problem is somewhat involved. If the equation of the line as well as the sign of the distance be known, there is but one answer: a single line parallel to the given line and on the determinate side of it. If the equation of the line is not given, one of two possible lines is clearly the answer, but which one? The transcendental theory prohibits two answers, and requires that one be selected in a prescribed manner, but does not furnish the data. If a student does as he is told and arbitrarily selects the coördinate system in some convenient way, he will obtain a unique and supposedly correct answer. But two different students may each assert that his own solution is unique and correct, while these may correspond to opposite determinations of sign.

4. Find the locus of a point such that the distance between the point and a given line is a given constant.

This is merely a catch question and has no answer, because the distance is either from the line to the point or from the point to the line and these are distinct. There is no single distance, unless it be zero, between the point and the line.

5. Find the locus of points equidistant from two given intersecting lines.

This problem is impossible unless the position of the origin be known, and should the origin be on one of the lines, the position of the axes must also be given. These data known, the answer is a single line.

6. Find the center of the inscribed circle of a given triangle.

This is but occasionally a point equidistant from the three sides. This being a point determined by the figure and not by the coördinate system, all special transcendental conventions are useless. The point equidistant from the three sides by the transcendental convention, may be tested algebraically or graphically and has roughly only one chance out of four of being the center of the inscribed circle.

7. Find the locus of points equidistant from two given parallel lines.

The locus consists of no points, unless the origin should fall within the two lines, or in certain cases upon one of the lines.

8. Given AB perpendicular to BC, find the distance from A to BC.

This problem is impossible unless the position of the origin be known. The distance may or may not be equal to the positive length AB.

9. Calling the locus of points one unit distant from a given line, the line one

unit distant from the given line, find the envelope of the lines one unit distant from the lines through the origin.

The envelope is not a circle, but half a circle.

10. Given a fixed point, (x_0, y_0), other than the origin. Any line through (x_0, y_0) will have an equation of the form, $(x - x_0) \cos \theta + (y - y_0) \sin \theta = 0$. Determine the character of the distance from any point, (x', y'), to the line, $(x - x_0) \cos \theta + (y - y_0) \sin \theta = 0$, as θ varies.

The distance is a periodic function of θ. If (x', y') is not collinear with the origin and (x_0, y_0), the distance has a finite discontinuity as $\tan \theta$ passes through the value $- x_0/y_0$. When (x', y') is on the line joining (x_0, y_0) and the origin, the distance is constantly positive or constantly negative, depending upon the order of the points $(0, 0)$, (x', y'), (x_0, y_0), on this common line.

The transcendental definition has the great advantage of emphasizing the fact that the sign of the distance from a line may be made of use in determining whether or not two points given by their coördinates are on the same or opposite sides of a given line, but in so far as it specifically determines a positive or negative sign, there seems to be no advantage derived. One might protest in behalf of the fundamental conventions of analytical geometry with regard to cöordinates. A reply is, however, obvious. The coördinates may be held to involve more than distances, they involve directions and this distinction is regularly observed at the start. The simpler geometric notions of distance are independent of the position of coördinate axes, and only suffer by special conventions attempting to relate them.

3. **Algebraic Determination.** As is usual in algebra one can regard distances as inherently either plus or minus as far as algebraic distinctions are concerned. It is possible to explain and emphasize the fact that in any particular example a fixed sign may be given to the expression, which sign is selected being clearly immaterial, and so the fact as to whether two points are on the same or opposite sides is readily determined. Mention may be made with regard to the convenient character of the origin as a reference point when it happens to be given. At the same time any complete answer to an exercise must take into consideration the possibility that the student has a free choice in sign and the solutions will be as numerous as the possible combinations of signs. This is just what is done throughout in algebra. For example, if $x^2 - 3x + 2 = 0$, be an equation proposed and it is desired to find the solution x intended by the proposer, the problem is not possible algebraically. One can only say that x must have been either 1 or 2, and more cannot be determined except possibly by other methods. This does not mean that the proposer had necessarily these two solutions in mind. He may have intended one or the other or both, but which possibility cannot be found by algebra alone if at all. Thus the locus of points at one unit's distance from a given line, may be viewed as a single line on one side, or one on the other side or as both lines. The only legitimate answer appears to be a pair of lines, with the understanding that either one alone might be the answer intended by special conventions. The algebraical determination has all the completeness of

the numerical determination, and all the flexibility of the transcendental. It has also its own objections, however. The algebraical determination always gives complete algebraic curves instead of portions as sometimes given by each of the other conventions. Its disadvantage is that plus a distance and minus a distance are not distinguished. Let us examine some instances.

1. Find the locus of a point such that the sum of its distances from two given points is a constant.

The solution is either an ellipse, an hyperbola, or a straight line, according to the relation of the constant to the distance between the two given points. By the numerical definition, the ellipse and part of the line correspond to sums, and the hyperbola and the rest of the line to differences, the two branches of the hyperbola, and the two rays of the line corresponding to the two choices of subtrahend and minuend.

2. Find the locus of a point such that the sum of its distances from the rectangular axes is equal to unity.

The locus is four complete straight lines, viz.,

$$x + y = 1, \qquad x - y = 1, \qquad -x + y = 1, \qquad -x - y = 1,$$

parts of the lines corresponding in the numerical definition to distances.

3. Find the center of the inscribed circle of a given triangle.

The three lines of the triangle have each two sides which are half planes. The two sides may be at once distinguished algebraically, if the opposite vertex be in each case compared with any point under consideration. Thus of the four points obtained as equidistant from the lines, that one which is simultaneously on the same side of each line as the opposite vertex is the center of the inscribed circle. If an origin be given not on any of the lines, or any other point be convenient, this point might be used as reference for all three lines if preferred.

Conclusion. No teacher or good student is badly misled by the transcendental conventions as to distances in the analytical geometry textbooks. He uses the convention when convenient and rejects it otherwise. The definitions there given are, however, baffling to the poor student, and either examples must be skilfully selected in the textbook or by the teacher to avoid trouble, or else the conventions which have been so carefully drilled into the memory of the unreasoning must be denied at just those moments when the pupil feels most helpless. It seems time to reject the transcendental concepts which are neither simple nor tenable, in behalf of either the obvious but difficult numerical determination or the simple but perhaps over-inclusive algebraic determination. The author realizes, however, that this cannot be a new question and it is one which must have often been examined by teacher and by textbook writer. A reply in defense of the accepted methods might prove of general interest.

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